Racah-Wigner algebra for q-deformed algebras

1992 J. Phys. A: Math. Gen. 25 1223

(http://iopscience.iop.org/0305-4470/25/5/026)

J. Phys. A: Math. Gen. 25 (1992) 1223-1235. Printed in the UK

Racah–Wigner algebra for q-deformed algebras

Cindy R Lienert and Philip H Butler

Physics Department, University of Canterbury, Christchurch, New Zealand

Received 18 June 1991, in final form 28 October 1991

Abstract. The concepts of vector coupling coefficients and recoupling coefficients are generalized to q-deformed algebras. Their properties under complex conjugation and permutation of irreps are derived. Relations betwen the coefficients, for example, the Racah backcoupling rule, are proved. We show that these properties may be used to recursively calculate the coupling and recoupling coefficients for all q-algebras. The $3jm$ and $6j$ symbols for $su(2)_q$ are used to illustrate the building-up method.

1. Introduction

In recent years there has been a great deal of interest in the q-deformations of Lie algebras sometimes known as 'quantum groups'. These structures were first uncovered in the study of Yang-Baxter equations (Kulish and Reshetikhin 1981, Sklyanin 1982). The classical Yang-Baxter equation is related to the Jacobi identity of a classical Lie algebra. The quantum equation forms a key element of algebraic structures which are one-parameter deformations of Lie algebras. Yang-Baxter type equations arise in statistical mechanics (Baxter 1982), conformal field theory (de Vega 1989, Witten 1990) and as the multiplication rule for braid groups (Akutsu and Wadati 1987), and so are of importance to physicists and mathematicians. Solutions of the Yang-Baxter equation, the R-matrices, may be based on coupling coefficients of q-deformed Lie algebras (Pasquier 1988, Kuniba 1990, Nomura 1989a, b, Hou *et al* 1990a).

The Racah-Wigner algebra of $\mathrm{su}(2)_q$ has been developed and the vector coupling coefficients (the $3jm$ symbols) and the recoupling coefficients (the $6j$ symbols) have been obtained and the corresponding R-matrices found (Kirillov and Reshetikhin 1988, Koelink and Koornwinder 1989, Groza *et al* 1990, Hou *et al* 1990b, Kachurik and Klimyk 1990, Nomura 1989a, b, Ruegg 1990). A few vector coupling coefficients have been calculated by Koh and Ma (1990) and Kuniba (1990) for the exceptional groups and by Ma (1990a, b) for $su(3)_q$. Reshetikhin (1987) gives some of the properties of general q-vector coupling coefficients and derives some of these for various groups. Jimbo (1985, 1987) has calculated the vector coupling coefficients for fundamental representations of some q-deformed algebras. However, some of the Racah-Wigner algebra for q-algebras has been missing.

The aim of this paper is to establish the Racah-Wigner algebra for the Drinfeld-Jimbo q-deformation of any compact Lie algebra and to show how it can be used to calculate coupling coefficients. We define and derive properties of the vector coupling coefficients and recoupling coefficients. We state and prove relations such as the Racah backcoupling rule. These properties may then be used to build up the coupling and recoupling coefficients by first finding primitive coefficients based on a chosen low-dimensional faithful irrep. This is a simple extension of the method used for $q = 1$ by

0305-4470/92/051223+13$04.50

Butler (1976), Searle and Butler (1988) and others. The building up method is illustrated in the final few sections by calculating the $3jm$ and $6j$ symbols of $\mathrm{su}(2)_q$.

2. General structure of q-deformed algebras

The q-deformation $\mathscr{L}_q$ of a Lie algebra $\mathscr{L}$ can be described in terms of the Chevalley basis. The algebra $\mathscr{L}_q$, with simple roots α_i and corresponding generators $X_i^\pm$ and H_i, has the deformed commutation relations (Drinfeld 1985, Jimbo 1985)

$$[H_i, H_j] = 0 \qquad [H_i, X_j^\pm] = \pm(\alpha_i\alpha_j)X_j^\pm \tag{1}$$

$$[X_i^+, X_j^-] = \delta_{ij}[H_i] \tag{2}$$

and Serre relations

$$\sum_k (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} q_i^{-k(n-k)/2}(X_i^\pm)^k X_j^\pm (X_i^\pm)^{n-k} = 0 \qquad q_i = q^{(\alpha_i\alpha_i)/2} \quad i \neq j \tag{3}$$

where for an operator or number x, we define

$$[x] \equiv \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \tag{4}$$

and for integers n and k

$$[n]! \equiv [n][n-1]\dots[1] \qquad \begin{bmatrix} n \\ k \end{bmatrix} \equiv \frac{[n]!}{[n-k]![k]!}. \tag{5}$$

The parameter q may take on arbitrary values except that in this paper it is assumed that q is not a root of unity. In the limit as $q \to 1$ the Lie algebra is retrieved. There are other choices of definition for the symbol $[x]$ and the comultiplication (Curtright *et al* 1991). In this paper, we have followed Reshetikhin (1987).

The q-deformed algebra is a Hopf algebra A (Abe 1980) having the following comutliplication $\Delta: A \to A \otimes A$

$$\Delta(X_i^\pm) = X_i^\pm \otimes q^{H_i/4} + q^{-H_i/4} \otimes X_i^\pm \qquad \Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i. \tag{6}$$

The Cartan subalgebra of $\mathscr{L}_q$ generated by $\{H_i\}$ is unchanged from that of $\mathscr{L}$ so the representation theory of the algebras are similar (Lusztig 1988, Rosso 1988), unless q is a root of unity when some of the representations are not completely reducible.

3. Vector coupling coefficients

An operator in $V^{\lambda_1} \otimes V^{\lambda_2}$ can be expressed in terms of operators in V^{λ_1} and V^{λ_2} by the comultiplication. Each operator may be realized by representations. A coupled representation $|(\lambda_1\lambda_2)r\lambda i\rangle$ can thus be expressed as a combination of the uncoupled representations $|\lambda_1 i_1\rangle|\lambda_2 i_2\rangle$. The vector coupling coefficients relate the two,

$$|\lambda_1 i_1\rangle|\lambda_2 i_2\rangle = \sum_{r\lambda i} |(\lambda_1\lambda_2)r\lambda i\rangle\, {}_q\langle r\lambda i|\lambda_1 i_1 \lambda_2 i_2\rangle \tag{7}$$

where r is a multiplicity label. The vector coupling coefficients form a unitary matrix

$$\sum_{i_1 i_2} {}_q\langle \lambda_1 i_1 \lambda_2 i_2 | r\lambda i\rangle\, {}_q\langle r'\lambda' i' | \lambda_1 i_1 \lambda_2 i_2\rangle = \delta_{rr'}\delta_{\lambda\lambda'}\delta_{ii'} \tag{8}$$

$$\sum_{r\lambda i} {}_q\langle \lambda_1 i_1 \lambda_2 i_2 | r\lambda i\rangle\, {}_q\langle r\lambda i | \lambda_1 i_1' \lambda_2 i_2'\rangle = \delta_{i_1 i_1'}\delta_{i_2 i_2'} \tag{9}$$

where ${}_q\langle\lambda_1 i_1 \lambda_2 i_2|r\lambda i\rangle$ denotes ${}_q\langle r\lambda i|\lambda_1 i_1 \lambda_2 i_2\rangle^*$. The action of an operator on a basis vector is described by a representation matrix

$$G|\lambda i\rangle = |\lambda i\rangle' = \sum_j |\lambda j\rangle \lambda(G)_{ji}. \tag{10}$$

The coupling coefficients reduce the product of representation matrices

$$\sum_{i_1 i_2 j_1 j_2} {}_q\langle r\lambda i|\lambda_1 i_1 \lambda_2 i_2\rangle \lambda_2(G)_{j_1 i_1} \lambda_2(G)_{j_2 i_2}\, {}_q\langle \lambda_1 j_1 \lambda_2 j_2|r'\lambda' i'\rangle = \delta_{\lambda\lambda'}\delta_{rr'}\lambda(\Delta(G))_{ii'}. \tag{11}$$

The product of representation matrices is not in general commutative because the comultiplication Δ is not invariant under interchange of the two spaces. However, it follows from (6) that

$${}_q\lambda_1(G)\,{}_q\lambda_2(G) = {}_{1/q}\lambda_2(G)\,{}_{1/q}\lambda_1(G) \tag{12}$$

where ${}_{1/q}\lambda(G)$ are representation matrices of $\mathscr{L}_{1/q}$. This property influences the symmetries of the vector coupling coefficients.

The complex conjugate matrix to $q_\lambda(G)$ is defined (Nomura 1990) so that

$$\sum_j {}_q\lambda(G)_{ij}\,{}_q\lambda(G)^*_{kj} = \sum_j {}_q\lambda(G)^*_{ji}\,{}_q\lambda(G)_{jk} = \delta_{ik} \tag{13}$$

and is denoted ${}_q\lambda(G)^*_{ij} \equiv {}_q\lambda(G)^{ij}$. The complex conjugate matrix is related to the matrix ${}_q\lambda^*(G)$ of the representation conjugate to λ. The unitary matrix relating the two is ${}_q(\lambda)_{ij}$, where

$${}_q\lambda^*(G)_{ij} = {}_q(\lambda)^{ik}\,{}_q\lambda(G)^{kl}\,{}_q(\lambda)_{lj} \tag{14}$$

and ${}_q(\lambda)^{ij} = {}_q(\lambda)^*_{ij}$.

With $\lambda_2 = \lambda_1^*$ in (11), λ is the identity representation, denoted 0, and it follows from (11) and (13) that

$${}_q(\lambda)_{lj} = {}_q\langle\lambda l \lambda^* j|00\rangle|\lambda|^{1/2} \tag{15}$$

where $|\lambda|$ is the q-dimension of λ. Reshetikhin (1988) gives the trivial vector coupling coefficient

$${}_q\langle\lambda\mu\lambda^* - \mu|00\rangle = \frac{q^{\rho(\mu)/2}}{|\lambda|^{1/2}}\phi_\mu \tag{16}$$

where ϕ is a phase and $\rho(\mu) = \frac{1}{2}\Sigma_{\alpha>0} H_\alpha(\mu)$, μ being a weight. Under complex conjugation we have ${}_q(\lambda)_{ij} = \{\lambda\}\,{}_{1/q}(\lambda^*)_{ji}$ so that $\{\lambda\}$ is the generalization of the $2j$ phase.

4. Properties of the coupling coefficients

The symmetry of the vector coupling coefficients under interchange of the first two irreps is not trivial, but rather from the property of the representation matrices (12) it follows that

$${}_q\langle\lambda_1 i_1 \lambda_2 i_2|r\lambda i\rangle = \{(12)\lambda_1\lambda_2\lambda^*\}_{rs}\,{}_{1/q}\langle\lambda_2 i_2 \lambda_1 i_1|s\lambda i\rangle \tag{17}$$

where $\{(12)\lambda_1\lambda_2\lambda^*\}_{rs}$ is the 3-j factor for this interchange. It is chosen to be $\{(12)\lambda_1\lambda_2\lambda^*\}_{rs} = \{\lambda_1\lambda_2\lambda^*\}\delta_{rs}$. Alternatively, the non-commutability of the representation matrices can be described by the braiding matrix or R-matrix

$$R_q^{\lambda_1\lambda_2}\,{}_q\lambda_1(G)\,{}_q\lambda_2(G) = {}_q\lambda_2(G)\,{}_q\lambda_1(G)R_q^{\lambda_1\lambda_2}. \tag{18}$$

The braiding matrix is a q-deformation of the permutation matrix mapping $V^{\lambda_1}\otimes V^{\lambda_2}$ into $V^{\lambda_2}\otimes V^{\lambda_1}$. The vector coupling coefficients are shown by Reshetikhin (1987) to have the following symmetry

$$(R_q^{\lambda_1\lambda_2})_{m_1'm_2'}^{m_1m_2}\ {}_q\langle\lambda_1 m_1\lambda_2 m_2|r\lambda m\rangle=\{\lambda_1\lambda_2\lambda^* r\}q^{\{c(\lambda)-c(\lambda_1)-c(\lambda_2)\}/2}\ {}_q\langle\lambda_2 m_2'\lambda_1 m_1'|r\lambda m\rangle \tag{19}$$

where $c(\lambda)$ is the quadratic Casimir operator acting on V^λ.

Another property of the coupling coefficients and R-matrices is the pentagonal relation (Reshetikhin 1987, Hou *et al* 1990)

$$\sum_{m_1m_2m'}(R_q^{\lambda_1\lambda})_{m_1'm}^{m_1m'}(R_q^{\lambda_2\lambda})_{m_2'm'}^{m_2m''}\ {}_q\langle\lambda_1 m_1\lambda_2 m_2|r\lambda_3 m_3\rangle={}_q\langle\lambda_1 m_1'\lambda_2 m_2'|r\lambda_3 m_3'\rangle(R_q^{\lambda_3\lambda})_{m_3m''}^{m_3'm}. \tag{20}$$

Complex conjugation of (11) and use of equations (13), (14) and (17) shows that the vector coupling coefficient and that obtained by replacing the irreps with their conjugates, are related by

$${}_q\langle\lambda_1 i_1\lambda_2 i_2|r\lambda i\rangle=A(\lambda_1\lambda_2\lambda)_{rs}\ {}_q(\lambda)^{ik}\ {}_{1/q}\langle s\lambda^* k|\lambda_1^* l_1\lambda_2^* l_2\rangle\ {}_q(\lambda_1)_{i_1l_1}\ {}_q(\lambda_2)_{i_2l_2} \tag{21}$$

where for most algebras $A_{rs}=\delta_{rs}$ (Butler 1975).

On interchanging λ_1 or λ_2 and λ_3, the vector coupling coefficients have the following symmetries

$$\sum_{j_2}{}_q\langle\lambda_1 j_1\lambda_2 j_2|r\lambda_3 j_3\rangle\ {}_q(\lambda_2)^{j_2l}=\frac{|\lambda_3|^{1/2}}{|\lambda_1|^{1/2}}\sum_s{}_q\langle s\lambda_1 j_1|\lambda_3 j_3\lambda_2^* l\rangle\{(13)\lambda_1\lambda_2\lambda_3^*\}_{rs} \tag{22}$$

$$\sum_{j_1}{}_q\langle\lambda_1 j_1\lambda_2 j_2|r\lambda_3 j_3\rangle\ {}_q(\lambda_1)^{j_1l}=\frac{|\lambda_3|^{1/2}}{|\lambda_2|^{1/2}}\sum_s{}_q\langle s\lambda_2 j_2|\lambda_1^* l\lambda_3 j_3\rangle\{(23)\lambda_1\lambda_2\lambda_3^*\}_{rs}. \tag{23}$$

To prove the first statement, we use the unitary property of the vector coupling coefficients to shift one of the coefficients in (11) to the right-hand side and multiply by $\lambda_2(G)^{j_2i_2}$ re-expressed using (14) to give

$$\begin{aligned}&{}_q\lambda_1(G)_{j_1i_1}\ {}_q\langle\lambda_1 j_1\lambda_2 j_2|r\lambda_3 j_3\rangle\\&\quad={}_q\langle\lambda_1 i_1\lambda_2 i_2|r\lambda_3 i_3\rangle\ {}_q\lambda_3(G)_{i_3j_3}\ {}_q(\lambda_2)_{j_2l_2}\ {}_q\lambda_2^*(G)_{m_2l_2}\ {}_q(\lambda_2)^{i_2m_2}.\end{aligned} \tag{24}$$

Substituting for $\lambda_3(G)\lambda_2^*(G)$ from (11) and rearranging

$$\begin{aligned}&{}_q\lambda_1(G)_{j_1i_1}\ {}_q\langle\lambda_1 j_1\lambda_2 j_2|r\lambda_3 j_3\rangle\ {}_q(\lambda_2)^{i_2m_2}\ {}_q\langle\lambda_3 j_3\lambda_2^* l_2|s\lambda_1' l_1\rangle\\&\quad={}_q\langle\lambda_1 i_1\lambda_2 i_2|r\lambda_3 i_3\rangle\ {}_q(\lambda_2)^{i_2m_2}\ {}_q\langle\lambda_3 i_3\lambda_2^* m_2|s\lambda_1' m_1\rangle\ {}_q\lambda_1'(G)_{m_1l_1}.\end{aligned} \tag{25}$$

Let $M_{j_1m_1}\equiv{}_q\langle\lambda_1 j_1\lambda_2 j_2|r\lambda_3 j_3\rangle\ {}_q(\lambda_2)^{i_2m_2}\ {}_q\langle\lambda_3 j_3\lambda_2^* l_2|s\lambda_1' l_1\rangle$ so that the last equation becomes

$$\lambda_1(G)_{j_1i_1}M_{j_1m_1}=M_{i_1l_1}\lambda_1'(G)_{m_1l_1}. \tag{26}$$

Schur's second lemma then states that either $M_{i_1l_1}$ is a multiple of the identity matrix and λ_1 and λ_1' are equivalent or $M_{i_1l_1}$ is the zero matrix, so that the result follows.

One could define a $q-3jm$ symbol in analogy with $3jm$ symbols for groups (Butler 1975) by

$${}_q\begin{pmatrix}\lambda_1&\lambda_2&\lambda_3\\i_1&i_2&i_3\end{pmatrix}_r=|\lambda_3|^{-1/2}\ {}_q(\lambda_3)^{i_3j_3}q^{\{2\rho(i_2)+\rho(i_1)\}/3}\ {}_q\langle\lambda_1 i_1\lambda_2 i_2|r\lambda_3^* j_3\rangle. \tag{27}$$

5. Recoupling coefficients

The recoupling coefficient arises when three representations are coupled to give a fourth. By considering coupling in two different orders the recoupling coefficient is shown to satisfy

$$\sum_{r_{12}\lambda_{12}l_{12}r} {}_q\langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1\lambda_2 l_2\rangle\, {}_q\langle r\lambda l|\lambda_{12}l_{12}\lambda_3 l_3\rangle$$

$$\times\, {}_q\langle\lambda_1,(\lambda_2\lambda_3)r_{23}\lambda_{23};\, r'\lambda|(\lambda_1\lambda_2)\lambda_3,\, r_{12}\lambda_{12};\, r\lambda\rangle$$

$$= \sum_{\lambda_{23}l_{23}} {}_q\langle r'\lambda l|\lambda_1 l_1\lambda_{23}l_{23}\rangle\, {}_q\langle r_{23}\lambda_{23}l_{23}|\lambda_2 l_2\lambda_3 l_3\rangle. \tag{28}$$

The $q-6j$ symbol is defined in terms of the recoupling coefficient by

$${}_q\begin{Bmatrix}\lambda_1 & \lambda_{23} & \lambda^* \\ \lambda_3^* & \lambda_{12} & \lambda_2\end{Bmatrix}_{rs_{23}s_{12}r'}$$

$$=\{|\lambda_{12}||\lambda_{23}|\}^{-1/2}\{\lambda_2\}\{\lambda_{12}\lambda_3\lambda^* r\}\{(23)\lambda_1\lambda_2\lambda_{12}^*\}_{r_{12}s_{12}}\{(123)\lambda_2\lambda_3\lambda_{23}^*\}_{r_{23}s_{23}}$$

$$\times\, {}_q\langle(\lambda_1\lambda_2)r_{12}\lambda_{12},\,\lambda_3;\, r\lambda|\lambda_1,(\lambda_2\lambda_3)r_{23}\lambda_{23};\, r'\lambda\rangle. \tag{29}$$

The symmetries of the recoupling coefficients are easily derived from those of the vector coupling coefficients and are very similar to the $q=1$ case.

$${}_q\langle(\lambda_1\lambda_2)r_{12}\lambda_{12},\,\lambda_3;\, r\lambda|\lambda_1,(\lambda_2\lambda_3)r_{23}\lambda_{23};\, s\lambda\rangle$$

$$=\{(13)\lambda_1\lambda_2\lambda_{12}^*\}_{r_{12}r'_{12}}\{(13)\lambda_{12}\lambda_3\lambda^*\}_{rr'}\{(13)\lambda_1\lambda_{23}\lambda^*\}_{ss'}\{\lambda_2\lambda_3\lambda_{23}^* r_{23}\}$$

$$\times\, {}_{1/q}\langle(\lambda^*\lambda_3)r'\lambda_{12}^*,\,\lambda_2;\, r'_{12}\lambda_1^*|\lambda^*,(\lambda_3\lambda_2)r_{23}\lambda_{23};\, s'\lambda_1^*\rangle \tag{30}$$

$$=\{(132)\lambda_1\lambda_2\lambda_{12}^*\}_{r_{12}r'_{12}}\{(13)\lambda_2\lambda_3\lambda_{23}^*\}_{r_{23}r'_{23}}\{(132)\lambda_1\lambda_{23}\lambda^*\}_{ss'}\{\lambda_{12}\lambda_3\lambda^* r\}$$

$$\times\, {}_q\langle(\lambda_{23}\lambda_3^*)r'_{23}\lambda_2,\,\lambda_{12}^*;\, r'_{12}\lambda_1^*|\lambda_{23},(\lambda_3^*\lambda_{12}^*)r\lambda^*;\, s'\lambda_1^*\rangle \tag{31}$$

$$=\{(23)\lambda_1\lambda_2\lambda_{12}^*\}_{r_{12}r'_{12}}\{(23)\lambda_1\lambda_{23}\lambda^*\}_{ss'}\{(13)\lambda_{12}\lambda_3\lambda^*\}_{rr'}\{(13)\lambda_2\lambda_3\lambda_{23}^*\}_{r_{23}r'_{23}}$$

$$\times\, {}_q\langle(\lambda_1^*\lambda)s'\lambda_{23},\,\lambda_3^*;\, r'_{23}\lambda_2|\lambda_1^*,(\lambda\lambda_3^*)r'\lambda_{12};\, r'_{12}\lambda_2\rangle. \tag{32}$$

Similar symmetries hold for other interchanges of the pairs (λ_1,λ_3^*), $(\lambda_{23},\lambda_{12})$ and (λ^*,λ_2). Complex conjugation involves a change from q to $1/q$, as for the vector coupling coefficients

$${}_q\langle(\lambda_1\lambda_2)r_{12}\lambda_{12},\,\lambda_3;\, r\lambda|\lambda_1,(\lambda_2\lambda_3)r_{23}\lambda_{23};\, r'\lambda\rangle$$

$$= {}_{1/q}\langle\lambda_1^*,(\lambda_2^*\lambda_3^*)r_{23}\lambda_{23}^*;\, r'\lambda^*|(\lambda_1^*\lambda_2^*)\lambda_3^*,\, r_{12}\lambda_{12}^*;\, r\lambda^*\rangle. \tag{33}$$

The recoupling coefficients also satisfy the orthogonality relation

$$\sum_{\lambda_2 r_{12}r_{23}} \{{}_q\langle(\lambda_1\lambda_2)r_{12}\lambda_{12},\,\lambda_3;\, r\lambda|\lambda_1,(\lambda_2\lambda_3)r_{23}\lambda_{23}s\lambda\rangle$$

$$\times\, {}_q\langle\lambda_1,(\lambda_2\lambda_3)r_{23}\lambda_{23};\, s'\lambda'|(\lambda_1\lambda_2)r_{12}\lambda_{12},\,\lambda_3;\, r'\lambda'\rangle\}$$

$$=\delta_{\lambda\lambda'}\delta_{rr'}\delta_{ss'}. \tag{34}$$

Two expressions relating recoupling coefficients are the Racah backcoupling rule and the Biedenharn–Elliott sum rule. The latter is unchanged for q-recoupling coefficients

$$\sum_t \{\{\lambda_1\lambda_{23}\lambda_{123}^* t\}\ {}_q\langle(\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3;r_{123}\lambda_{123}|\lambda_1,(\lambda_2\lambda_3)r_{23}\lambda_{23};t\lambda_{123}\rangle$$

$$\times\,{}_q\langle(\lambda_{23}\lambda_1)t\lambda_{123},\lambda_4;r\lambda|\lambda_{23},(\lambda_1\lambda_4)r_1\lambda_{14};s\lambda\rangle\}$$

$$=\sum_{r_{124}\lambda_{124}r'r'_{124}}\{\{\lambda_{12}\lambda_3\lambda_{123}^* r_{123}\}\{\lambda_1\lambda_2\lambda_{12}^* r_{12}\}\{\lambda_2\lambda_3\lambda_{23}^* r_{23}\}$$

$$\times\,{}_q\langle(\lambda_3\lambda_{12})r_{123}\lambda_{123},\lambda_4;r\lambda|\lambda_3,(\lambda_{12}\lambda_4)r_{124}\lambda_{124};r'\lambda\rangle$$

$$\times\,{}_q\langle(\lambda_2\lambda_1)r_{12}\lambda_{12},\lambda_4;r_{124}\lambda_{124}|\lambda_2,(\lambda_1\lambda_4)r_{14}\lambda_{14};r'_{124}\lambda_{124}\rangle$$

$$\times\,{}_q\langle\lambda_3,(\lambda_2\lambda_{14})r'_{124}\lambda_{124};r'\lambda|(\lambda_3\lambda_2)r_{23}\lambda_{23},\lambda_{14};s\lambda\rangle\}. \qquad (35)$$

The Racah backcoupling rule for recoupling coefficients is

$$q^{\{c(\lambda_1)+c(\lambda_2)+c(\mu_1)+c(\mu_2)\}/2}\ {}_q\langle(\lambda_1\mu_3)r_1\mu_2,\mu_1^*;r_3\lambda_3^*|\lambda_1,(\mu_3\mu_1^*)r_2\lambda_2;r_4\lambda_3^*\rangle$$

$$=\sum_{rr'\nu} q^{\{c(\nu)+c(\lambda_3)+c(\mu_3)\}/2}\{\mu_2\}\{\nu\}\{(13)\nu\mu_1^*\lambda_1^*\}_{rs}\{(132)\mu_2\lambda_2^*\nu\}_{r's'}$$

$$\times\{(23)\lambda_2\mu_3\mu_1^*\}_{r_2s_2}\{\lambda_1\lambda_2\lambda_3r_4\}$$

$$\times\,{}_q\langle(\lambda_2\nu)s'\mu_2,\mu_1;r_3\lambda_3^*|\lambda_2,(\nu\mu_1)s\lambda_1;r_4\lambda_3^*\rangle$$

$$\times\,{}_q\langle(\lambda_1\mu_3)r_1\mu_2,\lambda_2^*;s'\nu|\lambda_1,(\mu_3\lambda_2^*)s_2\mu_1;s\nu\rangle\frac{|\nu||\lambda_2|^{1/2}}{\{|\mu_1||\lambda_1||\mu_2|\}^{1/2}}. \qquad (36)$$

The backcoupling rule follows from the identity

$${}_q\langle(\lambda_1\mu_3)r_1\mu_2,\mu_1^*;r_3\lambda_3^*|\lambda_1(\mu_3\mu_1^*)r_2\lambda_2;r_4\lambda_3^*\rangle$$

$$=\sum_{\nu rr'}{}_q\langle(\lambda_1\mu_3)r_1\mu_2,\mu_1^*;r_3\lambda_3^*|(\mu_2\lambda_2^*)r'\nu,\mu_1^*;r\lambda_1\rangle$$

$$\times\,{}_q\langle(\mu_2\lambda_2^*)r'\nu,\mu_1^*;r\lambda_1|\lambda_1(\mu_3\mu_1^*)r_2\lambda_2;r_4\lambda_3^*\rangle$$

by expanding the right-hand side in terms of vector coupling coefficients, and using the symmetries proved for coupling coefficients to re-express four of the coupling coefficients

$${}_q\langle r'\nu p|\mu_2m_2\lambda_2^*k_2\rangle$$

$$=q^{\{c(\lambda_2)+c(\nu)-c(\mu_2)\}/2}(R_q^{\lambda_2\nu})_{n_2'p}^{n_2''p'}\ {}_q\langle\lambda_2n_2''\nu p'|\mu_2m_2\rangle$$

$$\times\frac{|\nu|^{1/2}}{|\mu_2|^{1/2}}\,{}_q(\lambda_2^*)^{k_2n_2'}\{\mu_2\}\{(132)\mu_2\lambda_2^*\nu\}_{r's'}$$

$${}_q\langle\nu p\mu_1^*l_1|r\lambda_1n_1\rangle$$

$$={}_q\langle r\nu p|\lambda_1n_1\mu_1m_1'\rangle\frac{|\nu|^{1/2}}{|\lambda_1|^{1/2}}\,{}_q(\mu_1^*)_{l_1m_1'}\{\nu\}\{(13)\nu\mu_1^*\lambda_1^*\}_{rs}$$

$${}_q\langle r_4\lambda_3^*k_3|\lambda_1n_1\lambda_2n_2\rangle$$

$$=q^{\{c(\lambda_3)-c(\lambda_1)-c(\lambda_2)\}/2}(R_q^{\lambda_2\lambda_1})_{n_2n_1}^{n_2'n_1'}\ {}_q\langle r_4\lambda_3^*k_3|\lambda_2n_2'\lambda_1n_1'\rangle\{\lambda_1\lambda_2\lambda_3r_4\}$$

$$
{}_q\langle r_2\lambda_2 n_2|\mu_3 m_3\mu_1^* l_1\rangle
= q^{\{c(\mu_3)-c(\mu_1)-c(\lambda_2)\}/2}(R_q^{\mu_1\lambda_2})^{m_1''n_2'}_{m_1'n_2}\,{}_q\langle r_2\mu_1 m_1''|\mu_3 m_3\lambda_2^* k_2'\rangle
\times\frac{|\lambda_2|^{1/2}}{|\mu_1|^{1/2}}\,{}_q(\mu_1)^{m_1'l_1}\,{}_q(\lambda_2^*)_{k_2'n_2'}\{(23)\lambda_2\mu_3\mu_1^*\}_{r_2s_2}.
$$

On application of the pentagonal relation, (20), the R-matrices cancel giving the result. The Biedenharn–Elliott sum rule follows in a similar way.

6. Method of calculation

The properties of the coupling and recoupling coefficients that have been proved in the previous sections can be used in their calculation. The building up method used in the $q=1$ case (Butler and Wybourne 1976, Butler 1976, Searle and Butler 1988) can be adapted readily for use for q-deformations of Lie algebras. All the irreps of the algebra can be obtained from the comultiplication of a faithful representation. The component irreps of the lowest dimension faithful representation are called primitive irreps. The primitive coupling coefficients are then defined to be the non-trivial coefficients containing a primitive irrep at least once.

The primitive recoupling coefficients can be calculated from their orthogonality (34) and by use of the Racah backcoupling rule (36). The Biedenharn–Elliott sum rule (35) may then be used to find recoupling coefficients with higher dimensional irreps. The primitive coupling coefficients are similarly found from their orthogonality properties (8) and (9). Both the primitive recoupling coefficients and vector coupling coefficients can then be used via the recoupling equation (28) to find the general coupling coefficients. This approach has been used to confirm the results obtained for some of the coupling coefficients of G_2 by Kuniba (1990). The final part of the paper illustrates this method by calculating the coupling and recoupling coefficients for $\mathrm{su}(2)_q$.

7. Structure of $\mathrm{su}(2)_q$

The results of the preceding sections are applied to the q-algebra $\mathrm{su}(2)_q$. This algebra has representations which are labelled by j and m as for su(2), the group of angular momentum. The irreps have the q-dimension

$$
|j|=[2j+1]. \tag{37}
$$

For $\mathrm{su}(2)_q$ the irreps are real and the multiplicities are all 1. From equation (16) the trivial vector coupling coefficient has the value

$$
{}_q\langle jjmm'|00\rangle=\frac{(-1)^{j-m}\delta_{m-m'}q^{m/2}}{\sqrt{[2j+1]}}. \tag{38}
$$

Using the definition of the $q-6j$ symbol, (29), the trivial $q-6j$ is thus

$$
{}_q\begin{Bmatrix} j_1 & j_2 & j_3\\ j_2 & j_1 & 0\end{Bmatrix}=\frac{(-1)^{j_1+j_2+j_3}}{\sqrt{[2j_1+1][2j_2+1]}}. \tag{39}
$$

From the symmetries of the recoupling coefficients, (31), (32) and (32), the $q-6j$ symbol has the symmetries

$$ {}_q\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = {}_q\begin{Bmatrix} j_1 & j_3 & j_2 \\ l_1 & l_3 & l_2 \end{Bmatrix} = \ldots \tag{40} $$

$$ = {}_q\begin{Bmatrix} j_1 & l_2 & l_3 \\ l_1 & j_2 & j_3 \end{Bmatrix} = \ldots \tag{41} $$

The $q-3jm$ symbol can be defined for $\mathrm{su}(2)_q$ (Nomura 1989, Hou *et al* 1990) by

$$ {}_q\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3}\{[2j_3+1]\}^{-1/2} q^{m_2-m_1/6}\, {}_q\langle j_1 m_1 j_2 m_2 | j_3 - m_3 \rangle. \tag{42} $$

From the definition of the $3jm$, (42), and the symmetries of the vector coupling coefficients (17), (22) and (23), it follows that the symmetries of the $\mathrm{su}(2)_q$ $3jm$ symbol are

$$ {}_q\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3}\, {}_{1/q}\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \ldots \tag{43} $$

$$ = (-1)^{j_1+j_2+j_3}\, {}_{1/q}\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \tag{44} $$

The recoupling equation (28) may be re-expressed in terms of $\mathrm{su}(2)_q$ $6j$ and $3jm$ symbols as

$$ {}_q\begin{Bmatrix} j_2 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} {}_q\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} $$

$$ = \sum_{n_1 n_2 n_3} (-1)^{l_1+l_2+l_3+n_1+n_2+n_3} q^{-(n_1+n_2+n_3)/6}\, {}_{1/q}\begin{pmatrix} j_1 & l_2 & l_3 \\ m_1 & n_2 & -n_3 \end{pmatrix} $$

$$ \times\, {}_{1/q}\begin{pmatrix} l_1 & j_2 & l_3 \\ -n_1 & m_2 & n_3 \end{pmatrix} {}_{1/q}\begin{pmatrix} l_1 & l_2 & j_3 \\ n_1 & -n_2 & m_3 \end{pmatrix}. \tag{45} $$

8. Calculation of $\mathrm{su}(2)_q$ $6j$ symbols

The primitive irrep for $\mathrm{su}(2)_q$ is that for which $j=\frac{1}{2}$ so that the primitive $6j$ symbols are of one of the two forms

$$ {}_q\begin{Bmatrix} a & \frac{1}{2} & a+\frac{1}{2} \\ b-\frac{1}{2} & c & b \end{Bmatrix} \qquad {}_q\begin{Bmatrix} a & \frac{1}{2} & a-\frac{1}{2} \\ b-\frac{1}{2} & c & b \end{Bmatrix}. \tag{46} $$

The calculation for $6j$ symbols of $\mathrm{su}(2)_q$ is carried out in exactly the same manner as for su(2) (Butler 1976). The orthogonality properties (34) give three equations in the primitive $6j$s. On combining the equations using symmetries we obtain the relation

$$ [2a+2][2b+1]\, {}_q\begin{Bmatrix} a & \frac{1}{2} & a+\frac{1}{2} \\ b-\frac{1}{2} & c & b \end{Bmatrix}^2 = [2a][2b-1]\, {}_q\begin{Bmatrix} a & \frac{1}{2} & a-\frac{1}{2} \\ b-\frac{1}{2} & c & b-1 \end{Bmatrix}^2 \tag{47} $$

and on iteration

$$\left\{\begin{matrix} a & \frac{1}{2} & a+\frac{1}{2} \\ b-\frac{1}{2} & c & b \end{matrix}\right\}_q^2 = \frac{[2a]![2b-1]![2a+2-x]![2b+1-x]!}{[2a-x]![2b-1-x]![2a+2]![2b+1]!} \times \left\{\begin{matrix} a-x/2 & \frac{1}{2} & a-x/2+\frac{1}{2} \\ b-x/2-\frac{1}{2} & c & b-x/2 \end{matrix}\right\}_q^2. \tag{48}$$

The boundary condition occurs when x has its maximum value satisfying the triangle conditions, $x=a-b-c$. Then the orthogonality condition, (34), gives for the boundary $6j$

$$\left\{\begin{matrix} \frac{1}{2}(a-b+c) & \frac{1}{2} & \frac{1}{2}(a-b+c-1) \\ \frac{1}{2}(-a+b+c-1) & c & \frac{1}{2}(-a+b+c) \end{matrix}\right\}_q^2 = \frac{1}{[a-b+c+2][-a+b+c+1]}. \tag{49}$$

Substituting back, cancelling terms and taking the square root

$$\left\{\begin{matrix} a & \frac{1}{2} & a+\frac{1}{2} \\ b-\frac{1}{2} & c & b \end{matrix}\right\}_q = \alpha_{abc}\left\{\frac{[a-b+c+1][-a+b+c]}{[2a+1][2a+2][2b][2b+1]}\right\}^{1/2} \tag{50}$$

where α_{abc} is a phase to be determined. Using (50) and the orthogonality condition (34) together with a q-number identity from Andrews (1976) gives the second primitive $6j$

$$\left\{\begin{matrix} a & \frac{1}{2} & a-\frac{1}{2} \\ b-\frac{1}{2} & c & b \end{matrix}\right\}_q = \beta_{abc}\left\{\frac{[a+b-c][a+b+c+1]}{[2a][2a+1][2b][2b+1]}\right\}^{1/2}. \tag{51}$$

The Racah backcoupling rule (36) together with the phase of the trivial $6j$ and the symmetries of the $6j$ symbols enable the phases α_{abc} and β_{abc} to be found

$$\alpha_{abc}=\beta_{abc}=(-1)^{a+b+c}. \tag{52}$$

The general $6j$ symbols are found from the primitive $6j$ symbols by using the Biedenharn-Elliott sum rule (35). Substituting the primitive $6j$, (50), into the right-hand side of equation (35) and simplifying using symmetries and the substitution

$$\left|\begin{matrix} a & b & c \\ p & r & s \end{matrix}\right|_q = \frac{[p+r-c]![a-r+s]![-p+b+s]!}{[a+r+s+1]![b+p+s+1]!} \times\{\Delta(abc)\Delta(aps)\Delta(bps)\Delta(prc)\}^{-1}\left\{\begin{matrix} a & b & c \\ p & r & s \end{matrix}\right\}_q \tag{53}$$

where

$$\Delta(abc)=\left\{\frac{[a+b-c]![a-b+c]![-a+b+c]!}{[a+b+c+1]!}\right\}^{1/2} \tag{54}$$

gives the recursion relation

$$[2s+1]\left|\begin{matrix} a & b & c \\ p & r & s \end{matrix}\right| = \left|\begin{matrix} a & b & c \\ p-\frac{1}{2} & r-\frac{1}{2} & s+\frac{1}{2} \end{matrix}\right| - \left|\begin{matrix} a & b & c \\ p-\frac{1}{2} & r-\frac{1}{2} & s-\frac{1}{2} \end{matrix}\right|. \tag{55}$$

On iterating, we obtain

$$\begin{vmatrix} a & b & c \\ p & r & s \end{vmatrix} = \sum_{x=0}^{X} (-1)^x \begin{bmatrix} X \\ x \end{bmatrix} \frac{[2s-x]!}{[2s+1+X-x]!} [2s+1+X-2x] \times \begin{vmatrix} a & b & c \\ p-X/2 & r-X/2 & s+X/2-x \end{vmatrix}. \tag{56}$$

This solution is verified by substituting into equation (55) and combining the two terms on the right-hand side using a q-number identity (Andrews 1976).

The $6j$ symbol on the right-hand side of (56) has a stretched form when X has the maximum value $X=p+r-c$. The $6j$ symbol related to this stretched form by summetry (40) can be found from equation (56) with $X=2A$

$$\begin{vmatrix} a & s' & B \\ A & A+B & b \end{vmatrix} = \sum_{y=0}^{2A} (-1)^y \begin{bmatrix} 2A \\ y \end{bmatrix} \frac{[2b-y]!}{[2b+1+2A-y]!} [2b+1+2A-2y] \times \begin{vmatrix} a & s' & B \\ 0 & B & b+A-y \end{vmatrix}. \tag{57}$$

The only non-zero term on the right is that for which $y=b+A-s'$. Substituting the value of the trivial $6j$, (39), and the definition (53) into (57) we obtain for the stretched $6j$

$$\begin{vmatrix} a & b & A+B \\ A & B & s' \end{vmatrix} = \frac{(-1)^{a+b+A+B}[b-A+s']!}{[A-b+s']![b+A-s']![-a+s'+B]!} \times \frac{[a+b+A+B+1]!}{[a-s'+b]![a+b-A-B]!}. \tag{58}$$

Substituting into (56), using symmetries and a further substitution of $y=p+r+c+x$ yields an explicit form for the $q-6j$ symbols

$$\begin{Bmatrix} a & b & c \\ p & r & s \end{Bmatrix}_q = \sum_y (-1)^y [a+b+c+1+2(p+r+s-y)] \times \Delta(abc)\Delta(ars)\Delta(pbs)\Delta(prc) \times \prod_{\text{pairs}} \left\{ \frac{[a+r+s+1]![a+r+s+2p-y]!}{[-a+r+s]![b+c+r+s+2p+1-y]!} \times \frac{1}{[y-a-r-s]![b+c+r+s-y]!} \right\} \tag{59}$$

where the product is over cyclic permutations of the pairs (a, p), (b, r) and (c, s).

The previous calculations of the $\mathrm{su}(2)_q$ $6j$ symbols have involved direct substitution of the explicit formulas for the $3jm$ symbols and performing the summations using the q-binomial coefficients product rule or hypergeometric series (Kirillov and Reshetikhin 1988, Hou *et al* 1990b, Kachurik and Klimyk 1990). Alternatively, properties of the q-Hahn polynomials have been used (Koelink and Koornwinder 1989).

This method constructs the $6j$ symbols using only their properties, which in turn are derived from the structure of the q-algebra. The summations performed are simpler, with no q factors involved at all. Kachurik and Klimyk (1990) find a recursion relation similar to (55) but do not use it to find the $6j$ symbols.

The algebraic expression above for the $su(2)_q$ $6j$ symbols has a different number and structure of terms than that previously known (Kirillov and Reshetikhin 1988, Koelink and Koorwinder 1989, Hou *et al* 1990). Both are obtained from forms of $6j$ symbols for su(2) by replacing n with $[n]$. Butler (1976) notes it is not surprising that there are different forms for the $6j$ symbols since these are related to the hypergoemetric series ${}_4F_3$ for which many expressions are known.

9. Calculation of $su(2)_q$ $3jm$ symbols

All the primitive $3jm$'s are related by symmetries to

$$ {}_q\begin{pmatrix} j+\frac{1}{2} & j & \frac{1}{2} \\ -m-\frac{1}{2} & m & \frac{1}{2} \end{pmatrix}. \tag{60}$$

The calculation of this symbol is carried out in the same manner as for su(2) (Butler 1976); however, there are explicit powers of q involved. To calculate the explicit form of this symbol, we use the orthogonality relations (8) and (9) and the definition of the $q-3jm$ (42). After iterating $j+m+1$ times, and choosing an appropriate phase we obtain

$$ {}_q\begin{pmatrix} j+\frac{1}{2} & \frac{1}{2} & j \\ -m-\frac{1}{2} & \frac{1}{2} & m \end{pmatrix} = \left\{\frac{[j+m+1]}{[2j+1][2j+2]}\right\}^{1/2} (-1)^{j+m} q^{-3j+5m-1/12}. \tag{61}$$

Again following Butler (1976), we obtain

$$\begin{aligned} {}_q\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= q^{m_2-m_1/6} q^{\{c(j_1)+c(j_2)-c(j_3)+2(j_1j_2+j_1m_2-j_2m_1)\}/4} \\ &\quad\times\{[j_1+m_1]![j_1-m_1]![j_2+m_2]![j_2-m_2]![j_3+m_3]![j_3-m_3]!\}^{1/2} \\ &\quad\times(-1)^{j_1-j_2-m_3}\Delta(j_1j_2j_3)\sum_x\left\{\frac{(-1)^x q^{-x(j_1+j_2+j_3+1)/2}}{[x]![j_1+j_2-j-x]![j_1-m_1-x]!}\right. \\ &\quad\left.\times\frac{1}{[j_2+m_2-x]![j-j_2+m_1+x]![j-j_1-m_2+x]!}\right\} \end{aligned} \tag{62}$$

where $c(j)=j(j+1)$.

These $3jm$ symbols for $su(2)_q$ have been calculated previously. Groza *et al* (1990) and Hou *et al* (1990b) use the method of highest weights where the symbols for $m=j$ are found and then used to find the general symbols. Koelink and Koornwinder (1989) use q-Hahn polynomials and hypergeometric series, while Ruegg (1990) defines a q-derivative and constructs an invariant which is used to find the vector coupling coefficients. The current calculation uses the general properties of the $q-3jm$ symbols and the values of the primitive $6j$ symbols for $su(2)_q$ to find the general form for the

$q-3jm$'s rather than working directly from the comultiplication and representation matrices of the generators.

10. Conclusions

The Racah–Wigner algebra for a q-deformed Lie algebra is somewhat different to the Rach–Wigner algebra for the corresponding group: there is an explicit dependence on the deformation parameter q in the vector coupling coefficients and recoupling coefficients and symmetry properties of the coefficients are complicated by additional q to $1/q$ interchanges. The Bidenharn–Elliott sum rule is unchanged from the $q=1$ case but the Racah backcoupling rule has explicit q factors. In spite of these differences, the building-up method for calculating the coupling and recoupling coefficients can be extended to q-algebras.

The present recursive calculation of $q-3jm$ symbols for $su(2)_q$ is as straight forward as those known (Hou *et al* 1990, Kirillov and Reshetikhin 1988, Groza *et al* 1990, Ruegg 1990) giving the same result when differences in the definition of q are taken into account. The derivation in this paper of the $q-6j$ symbols using the building-up method is far simpler and less dependent on the special properties of q-series than those based on Racah's method (Hou *et al* 1990, Kirillov and Reshetikhin 1988).

References

Abe E 1980 *Hopf Algebras (Cambridge Tracts in Mathematics 74)* (Cambridge: Cambridge University Press)

Akutsu Y and Wadati M 1987 *J. Phys. Soc. Japan* **56** 3039–51

Andrews G E 1976 *The Theory of Partitions (Encyclopedia of Mathematics and its Applications vol 2)* (Reading, MA: Addison-Wesley)

Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)

Butler P H 1975 *Phil. Trans. R. Soc.* **277** 545–85

—— 1976 *Int. J. Quantum Chem.* **10** 599–613

Butler P H and Wybourne B G 1976 *Int. J. Quantum Chem.* **10** 581–98

Curtright T L, Ghandour G I and Zachos C K 1991 *J. Math. Phys.* **32** 676

de Vega H J 1989 *Int. J. Mod. Phys.* A **44** 2371–464

Drinfeld V G 1985 *Sov. Math.-Dokl.* **32** 254

Groza V A, Kachurk I I and Klimyk A U 1990 *J. Math. Phys.* **31** 2769–80

Hou B Y, Hou B Y and Ma Z Q 1990a *Commum. Theor. Phys.* **13** 181–98

—— 1990b *Commum. Theor. Phys.* **13** 341–54

Jimbo M 1985 *Lett. Math. Phys.* **10** 63–9

—— 1987 *Commum. Math. Phys.* **102** 537

Kachurik I I and Klimyk A U 1990 *J. Phys. A: Math. Gen.* **23** 2717–28

Kirillov A N and Reshetikhin N Y 1988 Representations of the algebra $U_q(2)$, q-orthogonal polynomials and invariants of links *Preprint* LOMI St. Petersburg

Koelink H T and Koornwinder T H 1989 *Proc. Kon. Ned. Akad. van Wetensch* A **92** 443–56

Koh I G and Ma Z Q 1990 *Phys. Lett.* **234B** 480–6

Kulish P P and Reshetikhin N Y 1981 *Zapishki Nauch. Semin. LOMI* **101** 101–10 (Engl. Transl. 1983 *J. Sov. Math.* **23** 2435

Kuniba A 1990 *J. Phys. A: Math. Gen.* **23** 1349–62

Lusztig G 1988 *Adv. Math.* **70** 237–49

Ma Z Q 1990a *J. Math. Phys.* **31** 550–6

—— 1990b *J. Math. Phys.* **31** 3079–84

Nomura M 1989a *J. Phys. Soc. Japan* **58** 2694–704

—— 1989b *J. Math. Phys.* **30** 2397–405

—— 1990 *J. Phys. Soc. Japan* **59** 4260–71

Pasquier V 1988 *Commum. Math. Phys.* **118** 355–64
Reshetikhin N Y 1987 Quantized universal enveloping algebras, the Yang–Baxter equation and invariants of links *Preprints* LOMI St. Petersburg E-4-87 and E-17-7
Rosso M 1988 *Commum. Math. Phys.* **117** 581–94
Ruegg H 1990 *J. Math. Phys.* **31** 1085–7
Searle B G and Butler P H 1988 *J. Phys. A: Math. Gen.* **21** 3041–50
Sklyanin E K 1982 *Funkt. Anal. Pril.* **16** 27–34 (Engl. Transl. 1982 *Funct. Anal. Appl.* **16** 263
Witten E 1990 *Nucl. Phys.* B **330** 285–346